Lie Group Symmetry of Motion

## Symmetry of Motion

For Physically-based animation,

Motion is usually described by the differential equation (1)

\[
\dot{x}=F(x,u)
\] (1)

\dot{X}=F(x,u)

Physically possible motion is the solution of the equation.

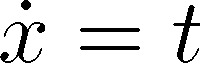
An important property from one solution x, with a group action g, we ca get another solution x\_a

\[
x_a=g_a(x)
\] (2)

x\_a=g\_a(x).

for example

,



We have

\[
x=0.5*t^2+c
\]

So the group action is

\[
g_a(x)=x+a
\]

For equation (1), the group action g\_a satisfy the symmetry property (2).

\[
\dot{g_a(x)}=F(u,g_a(x))
\] (3)

This provide us an idea about motion synthesis.Given an original motion m, and the corresponding group g, a new motion is generated by g(m).

## Local Motion Signature.

For every group G, we can find an function I(x) unchanged by the group action G,

\[
I(G(x))=I(x)
\]

I(x) are called local motion signature.

For mechanical system, Lie Group and Symmetry has important physically meaning.

I(x) corresponding to the Conservative Law, like energy or angular momentum.

## Controlled Symmetry

For motion synthesis, usually the desired motion is ma and original motion m is known, but the corresponding group action g\_a is not satisfied by differential equation.

For such situation, control input u is added, which modify the original equation to allow the designed G, this is called Controlled Symmetry.

Most dynamic motion can be modelled as an Lagrange System.

\[
L=K(\dot{q})-V(q)
\]

L=K(\dot(q)-V(q).

And the desired action G must keep the L invariant.

\[
L(G(q),G(\dot{q})=L(q,\dot{q})
\]

The original m is defined by the eural langrage equation

\[
\frac{d}{dt}\frac{\partial L}{\partial \dot{q}}-\frac{\partial L}{\partial q} = 0
\] (4)

\frac{d}{dt}\frac{\partial L}{\partial \dot{q}}-\frac{\partial L}{\partial q} = 0

The modified system is

\[
\frac{d}{dt}\frac{\partial L}{\partial G(\dot{q})}-\frac{\partial L}{G(\partial q)} = 0
\] (5)

\frac{d}{dt}\frac{\partial L}{\partial G(\dot{q})}-\frac{\partial L}{G(\partial q)} = 0

Which is equal the controlled dynamic system

\[
\frac{d}{dt}\frac{\partial L}{\partial \dot{q}}-\frac{\partial L}{\partial q} = u
\] (6)

\frac{d}{dt}\frac{\partial L}{\partial \dot{q}}-\frac{\partial L}{\partial q} = u

(5) and (6) are the equivalent equation, by comparing equation (5) and (6), we can get u

### Some Specific example of Symmetry and Control

### 1 Offset Action

\[
g_r(q,\dot{q})=(q+r,\dot{q})
\]

G\_r(x)=(q+r,\d{q})

Which keep speed, but modify the pos. thus keep the K but modify V

## 

u=\frac{\partial V(g\_r(q))-V(q)}{(\partial q)}

on phase space, if q is the horizontal axis, and \dot{q} is the vertical axis, this has the effect of moving the phase plot right and right.

## 2 Time Scalling.

\[
g_st(q,\dot{q})=(q,st*\dot{q})
\]

g\_st(q,dot{q})=(q,st\*dot{q})

we have

\[
u=(st^2-1) \frac{\partial V(q)}{(\partial q)}
\]

u=(st^2-1) \frac{\partial V(q)}{(\partial q)}

on phase space, this has the effect strength the phase plot in the vertical direction

## 3 energy scaling

For some system moving the the conservtime field with constant mass matrix.

The energy is preserved and different motion present different level of energy.

For such system, we have the

For such

\[
g_e(q,\dot{q})=(e^2*q,e*\dot{q})
\]

g\_e(q,\dot{q})=(e^2\*q,e\*\dot{q}).

U can be developed by applying the pos scaling and time scaling in a combined manner.

On phase plot, this has the effect enlarge the phase portrait.

## 4 time offset

Is q(t) is solution to f(q)

\[
g_{toff}(q(t) \dot{q}(t))=(q(t+toff),\dot{q}(t+toff))
\]

g\_{toff}(q(t) \dot{q}(t))=(q(t+toff),\dot{q}(t+toff))

For dynamic system, this seems obvious. And no control is need for such symmetry.

For system with limit circle, this g\_toff has a special effects like phase modification.

On phase plot, this has the effect rotate on the limit circle about an angle.